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OPERATIONAL RULES FOR A MIXED OPERATOR OF THE ERDÉLYI-KOBER TYPE

Yury Luchko

Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday

Abstract

In the paper, the machinery of the Mellin integral transform is applied to deduce and prove some operational relations for a general operator of the Erdélyi-Kober type. This integro-differential operator is a composition of a number of left-hand sided and right-hand sided Erdélyi-Kober derivatives and integrals. It is referred to in the paper as a mixed operator of the Erdélyi-Kober type.

For special values of parameters, the operator is reduced to some well known differential, integro-differential, or integral operators studied earlier by different authors. The differential operators of hyper-Bessel type, the Riemann-Liouville fractional derivative, the Caputo fractional derivative, and the multiple Erdélyi-Kober fractional derivatives and integrals are examples of its particular cases. In the general case however, the constructions suggested in the paper are new objects not yet well studied in the literature. The initial impulse to consider the operators presented in the paper arose while the author studied a problem to find scale-invariant solutions of some partial differential equations of fractional order: It turned out, that scale-invariant solutions of these partial differential equations of fractional order are described by ordinary differential equations of fractional order containing some particular cases of the mixed operator of Erdélyi-Kober type.

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1. Introduction

Recently, the operational calculus has become a powerful tool for solving many problems in different fields of analysis including ordinary and partial differential equations, integral equations and theory of special functions. Historically, the first operational calculi have been developed for some particular cases of the hyper-Bessel differential operator [16, 17, 32, 33]. Let us note that an operational calculus can be constructed both by using operational relations for a corresponding integral transform (say, the Laplace transform for the operator of differentiation) and by using the algebraic approach based on the convolution for the corresponding integral transform (Laplace convolution for the operator of differentiation) following the brilliant idea of Mikusiński. In this paper we restrict ourselves to the first approach; convolutions for the mixed Erdélyi-Kober operator and a Mikusiński type operational calculus for this operator will be presented elsewhere.

A very essential contribution to the operational calculi was done by Professor Dimovski who started to use systematically operational calculi to solve a number of applied problems as early as in the 70-ties. Among the other results, Professor Dimovski constructed an operational calculus for the hyper-Bessel differential operator [2, 4, 5], developed a non-local operational calculus [11] and a general theory of operational calculus for the abstract right inverse operators [6, 7, 9] and applied it to get representations of their multipliers and commutants [6, 8, 10]. Formulae for convolutions for some classical integral transforms have been obtained in Dimovski [3], Dimovski and Kalla [12], Dimovski and Kiryakova [13], [14], [15].

In fact, Professor Dimovski suggested a very powerful methodology for developing of operational calculi. He himself used his general scheme to construct operational relations and operational calculus of the Mikusiński type for the hyper-Bessel differential operator of the form

$$[Bf](x) = x^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} x \frac{d}{dx} \right) f(x), \quad \beta > 0, \gamma_i \in I\!R,\ i = 1, \dots, n. \quad (1)$$

The next step in generalization of Dimovski's results has been done by the author in [23], where operational calculi for some basic operators of fractional calculus including the Riemann-Liouville fractional derivative, the Caputo fractional derivative, the Erdélyi-Kober fractional derivative, and their compositions are developed. In [23] (see also [28, 29, 41]) an operational calculus for the general multiple Erdélyi-Kober fractional differentiation operator has been constructed for the first time. The multiple Erdélyi-Kober fractional differentiation operator has the following form:

$$[D_\mu f](x) = x^{-\mu} \prod_{i=1}^{n} \prod_{k=1}^{\eta_i} \left(k - \alpha_i - a_i\mu + a_i x \frac{d}{dx} \right) \left(\prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, \eta_i - a_i\mu} f \right)(x), \tag{2}$$

$$\mu > 0,\ a_i > 0,\ \alpha_i \in \mathbf{R},\ 1 \le i \le n,\ \eta_i = \begin{cases} [a_i\mu] + 1,\ a_i\mu \notin \mathbf{N}, \\ a_i\mu,\ a_i\mu \in \mathbf{N}, \end{cases}$$

where

$$[I_\beta^{\gamma,\delta} f](x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int_0^x (x^\beta - u^\beta)^{\delta-1} u^{\beta(\gamma+1)-1} f(u)du, \tag{3}$$

is the right-hand sided Erdélyi-Kober fractional integration operator. For one-parameter family of convolutions of the operator (3), one can see Kiryakova [19], [20, Ch.2], while convolutions for special classes of multiple Erdélyi-Kober fractional integrals (compositions of right-hand sided operators only) are proposed also in Luchko and Yakubovich [28, 29] and Kiryakova [20, Sect.5.4].

The most known particular cases of the multiple Erdélyi-Kober fractional differentiation operator are the hyper-Bessel differential operator (1) ($a_i = 1/\beta,\ \alpha_i = -\gamma_i,\ \eta_i = 1,\ 1 \le i \le n,\ \mu = \beta$) and the Riemann-Liouville fractional differential operator ($n = 1,\ a_1 = 1,\ \alpha_1 = 0$, and $\eta_1 = \eta$):

$$[D_{0+}^\mu f](x) \equiv \left(\frac{d}{dx} \right)^\eta (I_{0+}^{\eta-\mu} f)(x), \quad \eta = \begin{cases} [\mu] + 1,\ \mu \notin \mathbf{N}, \\ \mu,\ \mu \in \mathbf{N}, \end{cases} \tag{4}$$

where $[I_{0+}^\alpha f](x)$ is the right-hand sided Riemann-Liouville fractional integral operator

$$[I_{0+}^\alpha f](x) = \int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} f(u)du. \tag{5}$$

Despite the fact that the multiple Erdélyi-Kober fractional differentiation operator is a very general object containing many operators of Fractional

Calculus as its particular cases, some new operators of the Erdélyi-Kober type have appeared recently in applications. Even if these operators are compositions of the Erdélyi-Kober fractional integrals and derivatives, they cannot be represented in the form (2). In particular, in [25] (see also [1] and [18] for some different cases) a linear equation with the following operator $L_{\alpha,\beta,\gamma}$ has been introduced and solved:

$$[L_{\alpha,\beta,\gamma}f](x) = x^{\beta}[P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha} I_1^{-\beta,\beta} f](x), \tag{6}$$

where $I_1^{-\beta,\beta}$ is a particular case of the right-hand sided Erdélyi-Kober operator (3), whereas $P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha}$ is the left-hand sided Erdélyi-Kober fractional differential operator:

$$[P^{\tau,\alpha}_{\beta} f](x) = \prod_{j=0}^{\eta-1}(\tau + j - \frac{1}{\beta}x\frac{d}{dx})[K^{\tau+\alpha,\eta-\alpha}_{\beta} f](x), \tag{7}$$

$$\eta = \begin{cases} [\alpha]+1, \ \alpha \notin \mathbf{N}, \\ \alpha, \ \alpha \in \mathbf{N}, \end{cases}$$

the operator

$$[K^{\tau,\alpha}_{\beta} f](x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} \int_x^{\infty} (u^{\beta} - x^{\beta})^{\alpha-1} u^{-\beta(\tau+\alpha-1)-1} f(u)du, \tag{8}$$

being the left-hand sided Erdélyi-Kober fractional integral operator ($\beta > 0,\ \alpha > 0$).

The equation mentioned above has been deduced to describe the scale-invariant solutions of a model partial differential equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}, \qquad x > 0,\ t > 0, \tag{9}$$

where both fractional derivatives are defined in the Riemann-Liouville sense.

The operator (6) being a composition of the left- and right-hand sided Erdélyi-Kober operators is a new object in operational calculus: All known operational calculi for this kind of operators were constructed for some particular cases of the composition of the left-hand sided (or right-hand sided) Erdélyi-Kober operators.

In the paper, we consider a very general mixed operator of the Erdélyi-Kober type which is a composition of a number of left-hand sided and right-hand sided Erdélyi-Kober derivatives and integrals. In particular, the operator (6) is one of the particular cases of this general operator. The main focus in the paper is to demonstrate how the Mellin transform technique can be used to construct different representations of this operator, its particular cases, and operational properties. That is why we work in the next sections with a special space of functions that is optimized for the Mellin transform manipulations; the obtained constructions can be then proved in the classical spaces of functions as it has been done for example in [23, 24, 26, 27].

The remainder of the paper is organized as follows. In Section 2 we present some properties of the Mellin transform and the space of functions used in the further discussions. Section 3 is devoted to the generalized Obrechkoff-Stieltjes transform and to its particular cases. In Section 4, a general mixed operator of the Erdélyi-Kober type is introduced. Using the Mellin transform machinery operational rules for this operator are deduced. The convolutions and operational calculus for the mixed operator of the Erdélyi-Kober type will be presented elsewhere.

2. Mellin integral transform

An internal or intermediate language we use in the paper for presentation of functions, operators, and other objects is the language of the Mellin integral transform

$$f^*(s) = \mathcal{M}\{f(t);s\} = \int_0^{+\infty} f(t)t^{s-1}dt. \tag{10}$$

The inverse Mellin transform is given by the formula

$$f(x) = \mathcal{M}^{-1}\{f^*(s);x\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}ds,\ \gamma = \Re(s). \tag{11}$$

For the theory of the Mellin transform in different classical spaces of functions see for example [31, 41]. If we denote by $\rightarrow$ the correspondence between a function and its Mellin transform, then the following formulae of general type which we will use in further discussions can be easily proved

$$f(ax) \rightarrow a^{-s}f^*(s),\ a > 0; \tag{12}$$

$$x^p f(x) \rightarrow f^*(s+p); \tag{13}$$

$$f(x^p) \to \frac{1}{|p|} f^*(s/p),\ p \neq 0. \tag{14}$$

In the reference books [31] and [36] the very extensive lists of the Mellin transform formulae of special functions have been given. In the further discussions we need the following formulae (for definitions and properties of the special functions appearing in the formulae see e.g. [31], [36], [37], [41]):

$$e^{-x^p} \to \frac{1}{|p|}\Gamma(s/p),\ \Re(s/p) > 0, \tag{15}$$

where Γ is the Euler integral of the second kind

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx,\ \Re(s) > 0, \tag{16}$$

that is often reffered to as the Gamma-function;

$$\frac{(1-x^p)_+^{\alpha-1}}{\Gamma(\alpha)} \to \frac{\Gamma(s/p)}{|p|\Gamma(s/p+\alpha)},\ \Re(\alpha) > 0,\ \Re(s/p) > 0, \tag{17}$$

where $(x)_+ \equiv H(x)$, H is the Heaviside function;

$$\frac{(x^p-1)_+^{\alpha-1}}{\Gamma(\alpha)} \to \frac{\Gamma(1-\alpha-s/p)}{|p|\Gamma(1-s/p)},\ 0 < \Re(\alpha) < 1 - \Re(s/p); \tag{18}$$

$$\Gamma(\rho)(1+x)^{-\rho} \to \Gamma(s)\Gamma(\rho - s),\ 0 < \Re(s) < \Re(\rho); \tag{19}$$

$$\frac{1}{\pi(1-x)} \to \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s+1/2)\Gamma(1/2-s)},\ 0 < \Re(s) < 1; \tag{20}$$

$$K_\nu(2\sqrt{x}) \to \frac{1}{2}\Gamma(s+\nu/2)\Gamma(s-\nu/2),\ \Re(s) > |\Re(\nu)|/2, \tag{21}$$

K_ν is the Macdonald function;

$$e^x \mathrm{erfc}(\sqrt{x}) \to \frac{1}{\pi}\Gamma(s+1/2)\Gamma(s)\Gamma(1/2-s),\ 0 < \Re(s) < 1/2, \tag{22}$$

$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$, erf being the probability integral;

$$\Psi(a,b;x) \to \frac{1}{\Gamma(a)\Gamma(a-b+1)}\Gamma(s)\Gamma(s+1-b)\Gamma(a-s), \tag{23}$$

$$\max\{0, \Re(b-1)\} < \Re(s) < \Re(a),$$

Ψ is the Tricomi function;

$$e^{x/2}K_\nu\left(\frac{x}{2}\right) \to \frac{1}{\sqrt{\pi}}\cos(\pi\nu)\Gamma(s+\nu)\Gamma(s-\nu)\Gamma(1/2-s), \tag{24}$$

$$|\Re(\nu)| < \Re(s) < 1/2,$$

K_ν is the Macdonald function;

$$\pi 2^{\mu+1}\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)|1-x|^{\mu/2}P_\nu^\mu(\sqrt{x}) \to \tag{25}$$

$$\to \Gamma(s)\Gamma(s+1/2)\Gamma((1+\nu-\mu)/2-s)\Gamma(-(\mu+\nu)/2-s),$$

$$0 < \Re(s) < \min\{(1+\Re(\nu-\mu))/2, -\Re(\nu+\mu)/2\},$$

P_ν^μ is the Legendre function of the first kind;

$$J_\nu^2(\sqrt{x}) + Y_\nu^2(\sqrt{x}) \to \frac{2\cos(\nu\pi)}{\pi^{5/2}}\Gamma(s)\Gamma(s+\nu)\Gamma(s-\nu)\Gamma(1/2-s), \tag{26}$$

$$|\Re(\nu)| < \Re(s) < 1/2,$$

J_ν is the Bessel function of the first kind and Y_ν is the Neumann function;

$$H\begin{matrix} m,n \\ p,q \end{matrix}\left(z\,\middle|\,\begin{matrix}(\alpha_p,a_p)\\(\beta_q,b_q)\end{matrix}\right) \to \frac{\prod_{j=1}^m \Gamma(\beta_j+b_js)\prod_{j=1}^n \Gamma(1-\alpha_j-a_js)}{\prod_{j=n+1}^p \Gamma(\alpha_j+a_js)\prod_{j=m+1}^q \Gamma(1-\beta_j-b_js)} \tag{27}$$

$$-\min_{1\le j\le m}\Re(\beta_j)/b_j < \Re(s) < \min_{1\le j\le n}(1-\Re(\alpha_j))/a_j$$

and

1) $\sigma > 0$, or

2) $\sigma = 0$, $\delta\Re(s) < \frac{q-p}{2} - 1 + \Re\left(\sum_{j=1}^p \alpha_j - \sum_{j=1}^q \beta_j\right)$,

where

$$\sigma = \sum_{j=1}^n a_j - \sum_{j=n+1}^p a_j + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j,\ \delta = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j.$$

The **Fox's H-function** (see (27)) is defined by the following contour integral of the Mellin-Barnes type:

$$H\begin{matrix} m,n \\ p,q \end{matrix}\left(z\,\middle|\,\begin{matrix}(\alpha_p,a_p)\\(\beta_q,b_q)\end{matrix}\right) = H\begin{matrix} m,n \\ p,q \end{matrix}\left(z\,\middle|\,\begin{matrix}(\alpha,a)_{1,p}\\(\beta,b)_{1,q}\end{matrix}\right) = \frac{1}{2\pi i}\int_L \Phi(s)z^{-s}ds, \tag{28}$$

where $z \neq 0$, $0 \leq m \leq q$, $0 \leq n \leq p$, $\alpha_j \in \mathbf{C}$, $a_j > 0$, $1 \leq j \leq p$, $\beta_j \in \mathbf{C}$, $b_j > 0$, $1 \leq j \leq q$,

$$\Phi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + b_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s)}{\prod_{j=n+1}^{p} \Gamma(\alpha_j + a_j s) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j - b_j s)}, \tag{29}$$

an empty product, if it occurs, is taken to be one, and an infinite contour L separates all left poles $s = (-\beta_j - k)/b_j$, $j = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots$ of the numerator from the right ones $s = (1 - \alpha_j + k)/a_j$, $j = 1, 2, \ldots, n$, $k = 0, 1, 2, \ldots$ and under suitable conditions it may be one of the three types: $L_{-\infty}$, $L_{+\infty}$ or $L_{i\infty}$ (in particular, even a rectilinear line $L = (\gamma - i\infty, \gamma + i\infty)$). For the description of contours and detailed list of properties and particular cases of the H-function, see for example [36]. If all a_j, $j = 1, 2, \ldots, p$, and b_j, $j = 1, 2, \ldots, q$ are equal to 1, the Fox H-function (28) is called the **Meijer G-function**.

In this paper, we deal with integral transforms in a special space of functions $\mathcal{M}_\gamma^{-1}(L)$. The theory of this space can be found in [37, 39, 40, 41]. In particular, it turned out that this space of functions is very convenient for working with the Mellin convolution type transforms.

Definition 2.1. Let $\gamma \geq 0$. We denote by $\mathcal{M}_\gamma^{-1}(L)$ the space of functions representable by the inverse Mellin integral transforms

$$f(x) = \frac{1}{2\pi i} \int_\sigma f^*(s) x^{-s} ds, \ x > 0 \tag{30}$$

with functions f^* satisfying the condition $f^*(s)|s|^\gamma \in L(\sigma)$. Here $\sigma = \{s \in \mathbf{C} : \Re(s) = 1/2\}$ and $L(\sigma)$ denotes the space of Lebesgue-integrable on σ functions.

The space $\mathcal{M}_\gamma^{-1}(L)$ equipped with the norm

$$\|f\|_{\mathcal{M}_\gamma^{-1}(L)} := \frac{1}{2\pi} \int_\sigma |s^\gamma f^*(s) ds|$$

is a Banach space.

In the case $\gamma = 0$ the space $\mathcal{M}_\gamma^{-1}(L)$ will be denoted by $\mathcal{M}^{-1}(L)$.

We use also the following definition introduced in [39] and considered in details in [41] (see also [38]).

Definition 2.2. Denote by $\mathcal{K}$ the set of kernels $k : (0, \infty) \to \mathbf{R}$ for which the following conditions are fulfilled:

1) $k \in L(\epsilon, E)$ for any $\epsilon,\ E$, such that $0 < \epsilon < E < \infty$;
2) The integral

$$k^*(s) = \int_0^\infty k(u)u^{s-1}du,\ \Re(s) = \frac{1}{2} \tag{31}$$

is convergent and for almost all $\epsilon,\ E > 0$ and $t \in \mathbf{R}$ there exists a constant $C > 0$ such that

$$\left| \int_\epsilon^E k(u)u^{it-1/2}du \right| < C.$$

If for the kernel $k \in \mathcal{K}$ there exists the conjugate kernel $\hat{k} \in \mathcal{K}$ such that the equality

$$k^*(s)\hat{k}^*(1-s) = 1$$

holds almost everywhere on the line $\Re(s) = 1/2$, then we say that $k, \hat{k} \in \mathcal{K}^* \subset \mathcal{K}$.

To derive and prove our results, we use the following Parseval formula for the Mellin integral transform from [39] (see also [41]):

THEOREM 2.1. *Let $f \in \mathcal{M}^{-1}(L)$, $k \in \mathcal{K}$, k^* be given by* (31) *and f^* be determined by* (30). *Then the following Parseval formula takes place*

$$\int_0^\infty k(t/x)f(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_\sigma k^*(s)f^*(s)x^{-s}ds. \tag{32}$$

3. Generalized Obrechkoff-Stieltjes transform

We first consider a generalization of the Obrechkoff transform ([4], [35], [20, Ch.3], [41], [15]) in the form (see also [20, Ch.5], [34]):

$$[\mathcal{O}f](x) = \int_0^\infty H_{n,0}^{0,n}\left(\frac{x}{u}\middle| \begin{matrix}(\alpha, a)_{1,n} \\ -\end{matrix}\right) f(u)\frac{du}{u}, \quad x > 0. \tag{33}$$

By using the definition of H-function (28) and the representation of the Gamma-function (16) this transform can be rewritten in the following form

$$[\mathcal{O}f](x) = \int_0^\infty \Phi_n\left(x/u \mid (\alpha_i, a_i)_{1,n}\right) f(u)\frac{du}{u}, \tag{34}$$

where

$$\Phi(\tau|(\alpha_i, a_i)_{1,n}) = \frac{\tau^{(\alpha_n - 1)/a_n}}{a_n}\int_0^\infty \dots \int_0^\infty \exp\left\{-\sum_{i=1}^{n-1} u_i - \tau^{-\frac{1}{a_n}}\prod_{i=1}^{n-1} u_i^{-a_i/a_n}\right\}$$

$$\times \prod_{i=1}^{n-1} u_i^{-a_i \frac{1-\alpha_n}{a_n} - \alpha_i} du_1 \ldots du_{n-1}.$$

From Theorem 2.1 and using the asymptotic of the H-function from [41] and formula (27), we conclude that the generalized Obrechkoff transform (33) maps the space $\mathcal{M}^{-1}(L)$ into a subspace of $\mathcal{M}^{-1}(L)$ and the following representation takes place:

$$[\mathcal{O}f](x) = \frac{1}{2\pi i} \int_\sigma \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s) f^*(s) x^{-s} ds. \tag{35}$$

Another component we use for our generalization, the generalized Stieltjes transform, has been introduced in [41] in the form:

$$[\mathcal{S}_\beta^\alpha f](x) = \frac{x^{\frac{\alpha}{\beta}}}{\beta} \int_0^\infty \frac{f(u) u^{\frac{1-\alpha}{\beta}}}{x^{\frac{1}{\beta}} + u^{\frac{1}{\beta}}} \frac{du}{u}, \; x > 0, \; \beta > 0. \tag{36}$$

The case $\alpha = 0, \beta = 1$ of the operator (36) corresponds to the classical Stieltjes transform.

Theorem 2.1 and the formulae (19), (13), and (14) lead us to the following representation of the generalized Stieltjes transform on the space $\mathcal{M}^{-1}(L)$:

$$[\mathcal{S}_\beta^\alpha f](x) = \frac{1}{2\pi i} \int_\sigma \Gamma(1 - \alpha - \beta s) \Gamma(\alpha + \beta s) f^*(s) x^{-s} ds. \tag{37}$$

Based on the representations (35) and (37) of the generalized Obrechkoff and Stieltjes transforms, let us introduce the generalized Obrechkoff-Stieltjes transform that satisfies the following relation:

$$[\mathcal{OS}f](x) = \frac{1}{2\pi i} \int_\sigma \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s) \prod_{j=1}^{m} \Gamma(\beta_j + b_j s) f^*(s) x^{-s} ds. \tag{38}$$

Definition 3.1. The generalized Obrechkoff-Stieltjes transform is an integral transform of the Mellin convolution type with the Fox H-function of a special type as the kernel:

$$[\mathcal{OS}f](x) = \int_0^\infty H_{n,m}^{m,n} \left(\frac{x}{u} \middle| \begin{matrix} (\alpha, a)_{1,n} \\ (\beta, b)_{1,m} \end{matrix} \right) f(u) \frac{du}{u}, \; x > 0, \tag{39}$$

where

$$H^{m,n}_{n,m}\left(t\left|\begin{matrix}(\alpha,a)_{1,n}\\(\beta,b)_{1,m}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L \Phi(s)t^{-s}ds,$$

$$\Phi(s)=\prod_{j=1}^{n}\Gamma(1-\alpha_j-a_js)\prod_{j=1}^{m}\Gamma(\beta_j+b_js),$$

with the infinite contour L separating all left poles $s=(-\beta_j-k)/b_j$, $j=1,2,\dots,m$, $k=0,1,2,\dots$ of the expression $\prod_{j=1}^{n}\Gamma(1-\alpha_j-a_js)\prod_{j=1}^{m}\Gamma(\beta_j+b_js)$ from the right ones $s=(1-\alpha_j+k)/a_j$, $j=1,2,\dots,n$, $k=0,1,2,\dots$ and being of the type $L_{i\infty}$ (in particular, even a rectilinear line $L=(\gamma-i\infty,\gamma+i\infty)$). The contour $L_{i\infty}$ can be transformed into a left loop $L_{-\infty}$ under the conditions $\sum_{j=1}^{m}b_j>\sum_{j=1}^{n}a_j,\ |t|<+\infty$ or $\sum_{j=1}^{m}b_j=\sum_{j=1}^{n}a_j,\ |t|<1$ or into a right loop $L_{+\infty}$ under the conditions $\sum_{j=1}^{m}b_j<\sum_{j=1}^{n}a_j,\ |t|<+\infty$ or $\sum_{j=1}^{m}b_j=\sum_{j=1}^{n}a_j,\ |t|>1$.

Formula (27) for the Mellin transform of the H-function, Theorem 2.1, and the asymptotic of the H-function from [41] ensure the representation (38) of the generalized Obrechkoff-Stieltjes transform (39) on the space $\mathcal{M}^{-1}(L)$.

It follows from the relations (39), (35), and (37) that both the generalized Obrechkoff transform and the generalized Stieltjes transform are particular cases of the generalized Obrechkoff-Stieltjes transform (39). The formulae (21)–(26) lead to other particular cases different from the ones mentioned above (for the sake of reader's convenience some constant factors in the kernels of the integral transforms are omitted in several cases (see formulae (21)–(26)):

a) The modified Meijer's transform ($m=2$, $n=0$, $b_1=b_2=1$, $\beta_1=\nu/2$, $\beta_2=-\nu/2$):

$$[\mathcal{OS}f](x)=\int_0^\infty K_\nu(2\sqrt{\frac{x}{u}})f(u)\frac{du}{u},\ x>0; \tag{40}$$

b) The transform with the Macdonald function as the kernel ($m=2$, $n=1$, $b_1=b_2=a_1=1$, $\beta_1=\nu$, $\beta_2-\nu$, $\alpha_1=1/2$):

$$[\mathcal{OS}f](x)=\int_0^\infty e^{\frac{x}{2u}}K_\nu(\frac{x}{2u})f(u)\frac{du}{u},\ x>0; \tag{41}$$

c) The transform with the probability integral as the kernel ($m=2$, $n=1$, $b_1=b_2=a_1=1$, $\beta_1=1/2$, $\beta_2=0$, $\alpha_1=1/2$):

$$[\mathcal{OS}f](x)=\int_0^\infty e^{\frac{x}{u}}\mathrm{erfc}(\sqrt{\frac{x}{u}})f(u)\frac{du}{u},\ x>0; \tag{42}$$

d) The transform with the Tricomi function as the kernel ($m = 2$, $n = 1$, $b_1 = b_2 = a_1 = 1$, $\beta_1 = 0$, $\beta_2 = 1 - b$, $\alpha_1 = 1 - a$):

$$[\mathcal{OS}f](x) = \int_0^\infty \Psi(a;b;\frac{x}{u})f(u)\frac{du}{u},\ x > 0; \tag{43}$$

e) The transform with the Legendre function of the first kind as the kernel ($m = 2$, $n = 2$, $b_1 = b_2 = a_1 = a_2 = 1$, $\beta_1 = 0$, $\beta_2 = 1/2$, $\alpha_1 = 1 - (1 + \nu - \mu)/2$, $\alpha_2 = 1 + (\nu + \mu)/2$):

$$[\mathcal{OS}f](x) = \int_0^\infty |1 - \frac{x}{u}|^{\mu/2} P_\nu^\mu(\sqrt{\frac{x}{u}})f(u)\frac{du}{u},\ x > 0; \tag{44}$$

f) The transform with the sum of squares of the Bessel function and the Neumann function as the kernel ($m = 3$, $n = 1$, $b_1 = b_2 = b_3 = a_1 = 1$, $\beta_1 = 0$, $\beta_2 = \nu$, $\beta_3 = -\nu$, $\alpha_1 = 1/2$):

$$[\mathcal{OS}f](x) = \int_0^\infty (J_\nu^2(\sqrt{\frac{x}{u}}) + Y_\nu^2(\sqrt{\frac{x}{u}}))f(u)\frac{du}{u},\ x > 0. \tag{45}$$

Like the Obrechkoff transform (33), the generalized Obrechkoff-Stieltjes transform can be represented through a multiple integral with an exponential function and power multipliers as the kernel (see (34) for the representation of the Obrechkoff transform).

To illustrate the method, let us start with the modified Laplace transform that is a particular case of the Obrechkoff transform. Using the representation (16) of the Gamma-function and changing the order of integration in the double integral we get the following chain of equalities for a function f from the space $\mathcal{M}^{-1}(L)$ (see (30)):

$$\begin{aligned}[\mathcal{L}f](x) &= \frac{1}{2\pi i}\int_\sigma \Gamma(1-s)f^*(s)x^{-s}ds = \frac{1}{2\pi i}\int_\sigma\int_0^\infty e^{-t}t^{-s}dt\, f^*(s)x^{-s}ds \\ &= \int_0^\infty e^{-t}\frac{1}{2\pi i}\int_\sigma f^*(s)(xt)^{-s}ds\, dt = \int_0^\infty e^{-t}f(xt)\, dt.\end{aligned} \tag{46}$$

For the generalized Obrechkoff-Stieltjes transform represented in the form (38) the same method can be used if $\min\limits_{1\le i\le n}(1-\alpha_i)/a_i > \max\limits_{1\le i\le m} -\beta_j/b_i$:

$$[\mathcal{OS}f](x) = \frac{1}{2\pi i}\int_\sigma \prod_{j=1}^m \Gamma(\beta_j + b_j s)\prod_{j=1}^n \Gamma(1 - \alpha_j - a_j s)f^*(s)x^{-s}ds \tag{47}$$

$$= \frac{1}{2\pi i}\int_\sigma \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i\right)\prod_{i=1}^m u_i^{\beta_i+b_is-1} du_i \dots du_m$$

$$\times \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n v_i\right)\prod_{i=1}^n v_i^{-\alpha_i-a_is} dv_i \dots dv_n \; f^*(s)x^{-s}ds$$

$$= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i - \sum_{i=1}^n v_i\right)\prod_{i=1}^m u_i^{\beta_i-1}\prod_{i=1}^n v_i^{-\alpha_i} du_i \dots du_m \, dv_i \dots dv_n$$

$$\times \frac{1}{2\pi i}\int_\sigma f^*(s)\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right)^{-s} ds$$

$$= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^m u_i - \sum_{i=1}^n v_i\right)\prod_{i=1}^m u_i^{\beta_i-1}\prod_{i=1}^n v_i^{-\alpha_i}$$

$$\times f\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right) du_i \dots du_m \, dv_i \dots dv_n.$$

4. Mixed integro-differential operators of the Erdélyi-Kober type

In this section, an integro-differential operator of the Erdélyi-Kober type related to the generalized Obrechkoff-Stieltjes transform (39) is introduced. These two operators are connected by some operational relations. In fact, the generalized Obrechkoff-Stieltjes transform and the mixed integro-differential operator of the Erdélyi-Kober type we present in this section are very general and, in a certain sense, the limit generalizations of the well-known Laplace integral transform and the differential operator of the first order; they are connected each to other through the similar operational relations. The operational relations for the mixed operator of the Erdélyi-Kober type we consider in this section can be used to develop an operational calculus following the standard scheme.

In Introduction, several operators like the hyper-Bessel differential operator (1), the Riemann-Liouville fractional differential operator (4), the multiple Erdélyi-Kober fractional differential operator (2) and the Mikusiński type operational calculi constructed for these operators have been mentioned. But all these operators are nothing else than compositions of the right-hand sided (or left-hand sided) Erdélyi-Kober operators. To construct the operational relations for the generalized Obrechkoff-Stieltjes transform, operators of a new type are needed: the so called mixed operators of the Erdélyi-Kober type, i.e. the compositions of both the right-hand sided and the left-hand sided Erdélyi-Kober operators. A known particular case of this new general object is the Hilbert integral transform

$$[\mathcal{H}f](x) = \frac{1}{\pi}\int_0^\infty \frac{f(t)}{t-x}\,dt, \tag{48}$$

where the integral is considered in the sense of principal value.

In this section we start investigation of the general mixed operator of the Erdélyi-Kober type providing some of its properties and the operational relations connecting it with the generalized Obrechkoff-Stieltjes transform (39).

Definition 4.1. A mixed operator of the Erdélyi-Kober type is called the following integro-differential operator:

$$[\mathcal{L}_\eta f](x) = \begin{cases} x^\eta[\prod_{i=1}^m P_{1/b_i}^{\beta_i - b_i\eta, b_i\eta} \prod_{i=1}^n I_{1/a_i}^{-\alpha_i, a_i\eta} f](x), & \eta > 0, \\ x^\eta[\prod_{i=1}^n D_{1/a_i}^{-\alpha_i + a_i\eta, -a_i\eta} \prod_{i=1}^m K_{1/b_i}^{\beta_i, -b_i\eta} f](x), & \eta < 0, \end{cases} \tag{49}$$

where $a_i > 0$, $\alpha_i \in I\!R$, $i = 1,\ldots,n$; $b_i > 0$, $\beta_i \in I\!R$, $i = 1,\ldots,m$, $I_\beta^{\gamma,\delta}$, $K_\beta^{\tau,\alpha}$ are the right- and left-hand sided Erdélyi-Kober fractional integration operators (3) and (8), respectively (if $\delta = 0$ or $\alpha = 0$ these operators are defined as an identity operator), $P_\beta^{\tau,\alpha}$ is the left-hand sided Erdélyi-Kober fractional differential operator (7), and the right-hand sided Erdélyi-Kober fractional differential operator $D_\beta^{\gamma,\delta}$ is defined as follows:

$$[D_\beta^{\gamma,\delta} f](x) = \prod_{j=1}^{\eta}(\gamma + j + \frac{1}{\beta}x\frac{d}{dx})[I_\beta^{\gamma+\delta,\eta-\delta} f](x), \tag{50}$$

$$\eta = \begin{cases} [\delta] + 1, \ \delta \notin I\!N, \\ \delta, \ \delta \in I\!N. \end{cases}$$

Using the Parseval formula (32) (Theorem 2.1), formula (14) and the relations (17), (18) we arrive at the following representations for the Erdélyi-Kober operators in the space of functions $\mathcal{M}_\gamma^{-1}(L)$ (see e.g. [20], [37], [41]):

$$[I_\beta^{\gamma,\delta} f](x) = \int_\sigma \frac{\Gamma(1+\gamma - s/\beta)}{\Gamma(1+\gamma+\delta - s/\beta)} f^*(s) x^{-s} ds, \ \ \gamma,\ \beta > 0, \tag{51}$$

$$[K_\beta^{\tau,\alpha} f](x) = \int_\sigma \frac{\Gamma(\tau + s/\beta)}{\Gamma(\tau+\alpha+s/\beta)} f^*(s) x^{-s} ds, \quad \alpha,\ \beta > 0, \tag{52}$$

$$[D_\beta^{\gamma,\delta} f](x) = \int_\sigma \frac{\Gamma(1+\gamma+\delta - s/\beta)}{\Gamma(1+\gamma - s/\beta)} f^*(s) x^{-s} ds, \quad \gamma,\ \beta > 0, \tag{53}$$

$$(P_\beta^{\tau,\alpha} f)(x) = \int_\sigma \frac{\Gamma(\tau+\alpha+s/\beta)}{\Gamma(\tau + s/\beta)} f^*(s) x^{-s} ds, \quad \alpha,\ \beta > 0. \tag{54}$$

Taking into account the definition of the space $\mathcal{M}_\gamma^{-1}(L)$ and a corollary from the asymptotic Stirling formula in the form

$$|\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\pi|y|/2} \left[1 + O(1/y)\right], \ |y| \to \infty, \tag{55}$$

the relations (51)-(54) lead to the result, that the operator (51) maps the space $\mathcal{M}_\gamma^{-1}(L)$ onto $\mathcal{M}_{\gamma+\delta}^{-1}(L)$ and the operator (52) maps the space $\mathcal{M}_\gamma^{-1}(L)$ onto $\mathcal{M}_{\gamma+\alpha}^{-1}(L)$. For $\gamma \geq \delta$, the operator (53) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma-\delta}^{-1}(L)$. Finally, the operator (54) maps the space $\mathcal{M}_\gamma^{-1}(L)$ into $\mathcal{M}_{\gamma-\alpha}^{-1}(L)$ under the condition $\gamma > \alpha$.

The representations (51)-(54) and the mapping properties mentioned above result in the following very useful

THEOREM 4.1. *Under the condition*

$$\gamma + \eta \left(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i \right) > 0 \tag{56}$$

the mixed operator of the Erdélyi-Kober type (49) can be represented in the space $\mathcal{M}_\gamma^{-1}(L)$ in the following form:

$$[\mathcal{L}_\eta f](x) = \frac{x^\eta}{2\pi i} \int_\sigma \prod_{i=1}^n \frac{\Gamma(1-\alpha_i - a_i s)}{\Gamma(1-\alpha_i + a_i\eta - a_i s)} \prod_{i=1}^m \frac{\Gamma(\beta_i + b_i s)}{\Gamma(\beta_i - b_i \eta + b_i s)} f^*(s) x^{-s} ds. \tag{57}$$

Operator (57) maps the space $\mathcal{M}_\gamma^{-1}(L)$ on the space $\mathcal{M}^{-1}_{\gamma+\eta\left(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i\right)}(L)$.

The representation (57) is in fact the one that explains the idea behind the definition of the mixed operator of the Erdélyi-Kober type (49). If we denote the kernel of the generalized Obrechkoff-Stieltjes transform in the form (38) as

$$\Phi(s) = \prod_{j=1}^n \Gamma(1-\alpha_j - a_j s) \prod_{j=1}^m \Gamma(\beta_j + b_j s), \tag{58}$$

then the kernel of the mixed operator of the Erdélyi-Kober type (57) has the form

$$\frac{\Phi(s)}{\Phi(s-\eta)} = \prod_{i=1}^{n} \frac{\Gamma(1-\alpha_i - a_i s)}{\Gamma(1-\alpha_i + a_i\eta - a_i s)} \prod_{i=1}^{m} \frac{\Gamma(\beta_i + b_i s)}{\Gamma(\beta_i - b_i\eta + b_i s)}. \tag{59}$$

As shown in the next theorem, the representation (59) is the basis for the operational relation connecting the mixed Erdélyi-Kober type operator and the generalized Obrechkoff-Stieltjes transform.

THEOREM 4.2. *Let $f \in \mathcal{M}_\gamma^{-1}(L)$ and the condition (56) be fulfilled. Then the mixed Erdélyi-Kober type operator (49) and the generalized Obrechkoff-Stieltjes transform (39) are connected through the following operational relation:*

$$[\mathcal{OS}[\mathcal{L}_\eta f]](x) = x^\eta [\mathcal{OS} f](x), \tag{60}$$

that is, the generalized Obrechkoff-Stieltjes transform is a similarity which translates the mixed Erdélyi-Kober type operator into the operation of multiplication by the power function x^η.

P r o o f. According to Theorem 4.1, under the condition (56) the mixed Erdélyi-Kober type operator (49) can be represented in the space of functions $\mathcal{M}_\gamma^{-1}(L)$ in the following form:

$$[\mathcal{L}_\eta f](x) = \frac{x^\eta}{2\pi i} \int_\sigma \frac{\Phi(s)}{\Phi(s-\eta)} f^*(s) x^{-s} ds,$$

where $\Phi(s)$ is defined as in (58). Using the shift property of the Mellin transform (13) and the representation (38) of the generalized Obrechkoff-Stieltjes transform we have a simple chain of equalities:

$$[\mathcal{OS}[\mathcal{L}_\eta f]](x) = \frac{1}{2\pi i} \int_\sigma \Phi(s) [\mathcal{L}_\eta f]^*(s) x^{-s} ds$$

$$= \frac{1}{2\pi i} \int_\sigma \Phi(s) \frac{\Phi(s+\eta)}{\Phi(s)} f^*(s+\eta) x^{-s} ds$$

$$= \frac{1}{2\pi i} \int_\sigma \Phi(s+\eta) f^*(s+\eta) x^{-s} ds = \frac{x^\eta}{2\pi i} \int_\sigma \Phi(s) f^*(s) x^{-s} ds = x^\eta [\mathcal{OS} f](x).$$

■

Operational relation (60) can be used to solve a class of linear integro-differential equations containing the operator (49) of the type

$$\sum_{i=0}^{n} a_i[\mathcal{L}_\eta^i y](x) = f(x), \tag{61}$$

where $a_i,\ i = 0, \ldots, n$ are some numerical coefficients, $\mathcal{L}_\eta^i$ means a composition of i operators $\mathcal{L}_\eta$, and $\mathcal{L}_\eta^0$ is an identity operator: $\mathcal{L}_\eta^0 \equiv E$.

One example of the equation (61) (with $n = 1$) is the equation

$$y(x) - \lambda x^\eta [P_{\eta/\alpha}^{1+\gamma-\alpha,\alpha} I_1^{-\eta,\eta} y](x) = \sum_{j=1}^{n} c_j x^{2\eta-j},\ \eta > \alpha,\ \lambda,\ c_j \in I\!R,\ j = 1, \ldots, n,$$

that was deduced and solved in [25] to obtain the scale-invariant solutions of the model partial differential equation of fractional order (9). In the equation, the mixed operator of the Erdélyi-Kober type has the following form:

$$[\mathcal{L}_\eta f](x) = x^\eta [P_{\eta/\alpha}^{1+\gamma-\alpha,\alpha} I_1^{-\eta,\eta} f](x).$$

The operator $\mathcal{L}_\eta$ is a composition of the integral operator $I_1^{-\eta,\eta}$ of order η and the differential operator $P_{\eta/\alpha}^{1+\gamma-\alpha,\alpha}$ of order α. Because $\eta > \alpha$, it is an "integral" operator and therefore no initial conditions are required.

In the general situation, the mixed Erdélyi-Kober type operator (49) is an "integral" operator for $\eta > 0$ if $\sum_{i=1}^n a_i > \sum_{i=1}^m b_i$ and for $\eta < 0$ if $\sum_{i=1}^n a_i < \sum_{i=1}^m b_i$. If $\sum_{i=1}^n a_i < \sum_{i=1}^m b_i$ for $\eta > 0$ or $\sum_{i=1}^n a_i > \sum_{i=1}^m b_i$ for $\eta < 0$ the operator (49) is a "differential" one. Finally, if $\sum_{i=1}^n a_i = \sum_{i=1}^m b_i$ the mixed Erdélyi-Kober type operator (49) can be considered to be a generalization of the Hilbert transform (an example will be presented at the end of the section).

If the mixed Erdélyi-Kober type operator (49) is of the "integral type" ($\sum_{i=1}^n a_i > \sum_{i=1}^m b_i$ if $\eta > 0$ or $\sum_{i=1}^n a_i < \sum_{i=1}^m b_i$ if $\eta < 0$) the inverse operator of "differential type" is defined as follows:

$$[\mathcal{D}_\eta f](x) = \begin{cases} x^{-\eta}[\prod_{i=1}^n D_{1/a_i}^{-\alpha_i - a_i\eta, a_i\eta} \prod_{i=1}^m K_{1/b_i}^{\beta_i, b_i\eta} f](x), & \eta > 0, \\ x^{-\eta}[\prod_{i=1}^m P_{1/b_i}^{\beta_i + b_i\eta, -b_i\eta} \prod_{i=1}^n I_{1/a_i}^{-\alpha_i, -a_i\eta} f](x), & \eta < 0. \end{cases} \tag{62}$$

Using the properties of the Erdélyi-Kober operators (see e.g. [23, 37, 41]), it can be easy shown that the operator $\mathcal{L}_\eta$ is a right inverse to the operator $\mathcal{D}_\eta$, i.e.

$$[\mathcal{D}_\eta \mathcal{L}_\eta f](x) = f(x).$$

In the general case however, the operator $\mathcal{L}_\eta$ is not a left inverse to the operator $\mathcal{D}_\eta$. In particular, if $\eta > 0$ and $m = 0$, the operators $\mathcal{L}_\eta$ and $\mathcal{D}_\eta$ are the so called multiple Erdélyi-Kober fractional integration and differentiation operators (see Luchko [23], Luchko and Yakubovich [30]) given by:

$$[\mathcal{L}_\eta f](x) = x^\eta [\prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, a_i\eta} f](x), \tag{63}$$

$$[\mathcal{D}_\eta f](x) = x^{-\eta} [\prod_{i=1}^{n} D_{1/a_i}^{-\alpha_i - a_i\eta, a_i\eta} f](x), \tag{64}$$

and especially for $\eta = 1$ one gets the Gelfond-Leontiev integration and differentiation operators with respect to the multi-index Mittag-Leffler function $E_{(a_i),(1-\alpha_i)}(x)$, Kiryakova [21], [22].

It has been proven in [23] that for the operators (63) and (64) the following relation takes place in a suitable space of functions and under certain conditions:

$$[\mathcal{L}_\eta \mathcal{D}_\eta y](x) = y(x) - \sum_{i=1}^{n} \sum_{k=1}^{\eta_i} C_{ik} \left[\lim_{x\to 0} (A_{ik} y)(x) \right] x^{\eta - \frac{k-\alpha_i}{a_i}}, \tag{65}$$

where

$$C_{ik} = \frac{\prod_{j=i+1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)) \prod_{j=1}^{i-1} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i) + \eta_j)}{\prod_{j=1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i) + a_j\eta)},$$

$$(A_{ik} y)(x) = x^{-\eta + \frac{k-\alpha_i}{a_i}} \prod_{j=1}^{\eta_i - k} \left(k + j - \alpha_i - a_i\eta + a_i x \frac{d}{dx} \right)$$

$$\times \prod_{l=i+1}^{n} \prod_{j=1}^{\eta_i} \left(j - \alpha_l - a_l\eta + a_l x \frac{d}{dx} \right) \left(\prod_{j=1}^{n} I_{1/a_j}^{-\alpha_j, \eta_j - a_j\mu} y \right)(x),$$

$$\eta_i = \begin{cases} [a_i\eta] + 1, a_i\eta \notin I\!N, \\ a_i\eta, \ a_i\eta \in I\!N. \end{cases}$$

Similar, but even more complicated formula is valid in the general case of the mixed operator of the Erdélyi-Kober type (49) and its inverse differential operator (62).

REMARK 4.1. In Kiryakova [20], a more general definition and treatise of the multiple Erdélyi-Kober fractional integrals and derivatives has been presented. There, a multiple Erdélyi-Kober fractional integral is understood as follows:

$$[\mathcal{I}f](x) = x^{\beta_0}[I^{(\gamma_i),(\delta_i)}_{(\beta_i),n} f](x) := x^{\beta_0} \left(\prod_{i=1}^{n} I^{\gamma_i,\delta_i}_{\beta_i} f \right)(x) \tag{66}$$

$$= x^{\beta_0} \int_0^1 H^{n,0}_{n,n} \left[t \middle| \begin{array}{c} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i}) \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i}) \end{array} \right] f(xt)dt \ ,$$

where $\gamma_i \in \mathbf{R}$, $\delta_i \geq 0$, $\beta_i > 0$, $i = 1, \ldots, n, \beta_0 > 0$ and $\delta_i\beta_i > 0$, $i = 1, \ldots, n$ can be different, not obligatory all equal to $\beta_0 := \eta$, as it is in (63): $(a_i\eta) \times (1/a_i) = \eta, i = 1, \ldots, n$. However, the operator (66) does not in general possess the operational property (60) because its kernel cannot be represented in the form (59), the only exception being the case of the operator (63).

For the operator (62) the following Cauchy boundary value problem can be formulated:

$$\sum_{i=0}^{n} a_i[\mathcal{D}^i_\eta y](x) = f(x), \tag{67}$$

$$[F\mathcal{D}^k_\eta y](x) = \gamma_k(x), \ k = 0, 1, \ldots, n-1, \ \gamma_k(x) \in \ker \mathcal{D}_\eta,$$

where $F = \mathrm{Id} - \mathcal{L}_\eta\mathcal{D}_\eta$ is the projector of the operator $\mathcal{L}_\eta$. One example of the projector for a particular case of the operator $\mathcal{D}_\eta$, the multiple Erdélyi-Kober fractional differentiation operator, is given in the formula (65).

Following the standard scheme, the Cauchy boundary value problem (67) can be solved by applying the generalized Obrechkoff-Stieltjes transform (Theorem 4.2 and its proof remains valid in the space $\mathcal{M}^{-1}_\gamma(L)$ if we replace η with $-\eta$ and $\mathcal{L}_\eta$ with $\mathcal{D}_\eta$ thus resulting in the relation $[\mathcal{OS}[\mathcal{D}_\eta f]](x) = x^{-\eta}[\mathcal{OS}f](x)$; in the suitable "classical" spaces of functions the right-hand side of the last relation will additionally include some terms that correspond to the initial conditions of the Cauchy boundary value problem (67) (see [23, 24, 41])).

Particular cases of the multiple Erdélyi-Kober fractional differentiation operator are both the hyper-Bessel differential operator (1) (with $a_i = 1/\beta$, $\alpha_i = -\gamma_i$, $\eta_i = 1$, $1 \leq i \leq n$, $\eta = \beta$) and the Riemann-Liouville fractional differential operator (4) $\Big($with $n = 1$, $a_1 = 1$, $\alpha_1 = 0$, and $\eta_1 = \begin{cases} [\eta] + 1, \ \eta \notin I\!N \\ \eta, \ \eta \in I\!N \end{cases}\Big)$.

In the case of the Riemann-Liouville fractional differentiation operator (4) the representation (65) has the form

$$[\mathcal{L}_\eta \mathcal{D}_\eta y](x) = y(x) - \sum_{k=1}^{\eta_1} \frac{x^{\eta-k}}{\Gamma(\eta-k+1)} \lim_{x\to 0}(D_{0+}^{\eta-k} y)(x). \qquad (68)$$

For the hyper-Bessel differential operator (1) the formula (65) reduces to

$$[\mathcal{L}_\eta \mathcal{D}_\eta y](x) = y(x) - \sum_{i=1}^{n} x^{-\beta\gamma_i} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} \qquad (69)$$

$$\times \lim_{x\to 0} \left(x^{\beta\gamma_i} \prod_{j=i+1}^{n} (\beta\gamma_j + x\frac{d}{dx}) y(x) \right), \quad \gamma_1 < \gamma_2 < \cdots < \gamma_n < \gamma_1 + 1,$$

as established in Kiryakova [20], Th. 3.9.6, (3.9.29), p. 180-181.

Let us finally consider some particular cases of the mixed operator of the Erdélyi-Kober type (49).

Example 4.1. The simplest particular cases of the generalized Obrechkoff-Stieltjes transform (39) are so called modified Borel-Dzrbasjan's transforms [14, 23, 41, 20]:

$$[BD_+ f](x) = \int_0^\infty b^{-1} \left(\frac{x}{t}\right)^{\beta/b} \exp\left[-\left(\frac{x}{t}\right)^{1/b} \right] f(t) \frac{dt}{t}, \qquad (70)$$

$$[BD_- f](x) = \int_0^\infty a^{-1} \left(\frac{x}{t}\right)^{(\alpha-1)/a} \exp\left[-\left(\frac{x}{t}\right)^{-1/a} \right] f(t) \frac{dt}{t}. \qquad (71)$$

Relations (15), (13) and the Parseval formula (32) lead to the following representations of the modified Borel-Dzrbasjan's transforms:

$$[BD_+ f](x) = \frac{1}{2\pi i} \int_\sigma \Gamma(\beta + bs) f^*(s) x^{-s} ds, \qquad (72)$$

$$[BD_- f](x) = \frac{1}{2\pi i} \int_\sigma \Gamma(1 - \alpha - as) f^*(s) x^{-s} ds. \qquad (73)$$

The mixed operators of the Erdélyi-Kober type (49) corresponding to the modified Borel-Dzrbasjan transforms (70), (71) have the form (74) for the transform (72) and the form (75) for the transform (73), respectively:

$$[\mathcal{L}_\eta f](x) = x^\eta \frac{1}{2\pi i} \int_\sigma \frac{\Gamma(\beta + bs)}{\Gamma(\beta - b\eta + bs)} f^*(s) x^{-s} ds \qquad (74)$$

$$= \begin{cases} x^\eta (K_{1/b}^{\beta,-b\eta} f)(x), \ \eta < 0; \\ x^\eta (P_{1/b}^{\beta-b\eta,b\eta} f)(x), \ \eta > 0, \end{cases}$$

$$[\mathcal{L}_\eta f](x) = x^\eta \frac{1}{2\pi i} \int_\sigma \frac{\Gamma(1-\alpha-as)}{\Gamma(1-\alpha+a\eta-as)} f^*(s) x^{-s} ds \tag{75}$$

$$= \begin{cases} x^\eta (I_{1/a}^{-\alpha,a\eta} f)(x), \ \eta > 0; \\ x^\eta (D_{1/a}^{a\eta-\alpha,-a\eta} f)(x), \ \eta < 0, \end{cases}$$

where $I_\beta^{\gamma,\delta}, K_\beta^{\tau,\alpha}, D_\beta^{\gamma,\delta}, P_\beta^{\tau,\alpha}$ are the Erdélyi-Kober fractional integrals and derivatives.

Example 4.2. The mixed operators of the Erdélyi-Kober type corresponding to the generalized Obrechkoff transform (33) are nothing else than the multiple Erdélyi-Kober operators (63), (64):

$$[\mathcal{L}_\eta f](x) = \frac{x^\eta}{2\pi i} \int_\sigma \frac{\prod_{j=1}^n \Gamma(1-\alpha_j-a_j s)}{\prod_{j=1}^n \Gamma(1-\alpha_j - a_j(s-\eta))} f^*(s) x^{-s} ds \tag{76}$$

$$= \begin{cases} x^\eta \left(\prod_{j=1}^n I_{1/a_j}^{-\alpha_j,a_j\eta} f \right)(x), \ \eta > 0, \\ x^\eta \left(\prod_{j=1}^n D_{1/a_j}^{-\alpha_j+a_j\eta,-a_j\eta} f \right)(x), \ \eta < 0. \end{cases}$$

In particular, if $\eta = -\beta < 0, a_j = -1/\eta = 1/\beta, 1 \le j \le n$, and $-\alpha_j = \gamma_j, 1 \le j \le n$, then the operator (76) is reduced to the hyper-Bessel operator differential operator

$$[\mathcal{L}_\eta f](x) = x^{-\beta} \left(\prod_{j=1}^n D_\beta^{\gamma_j - 1,1} f \right)(x) = x^{-\beta} \prod_{j=1}^n \left(\gamma_j + \frac{1}{\beta} x \frac{d}{dx} \right) f(x). \tag{77}$$

Example 4.3. Due to the formula (37), the mixed operator of the Erdélyi-Kober type corresponding to the generalized Stieltjes transform (36) can be written in the form:

$$[\mathcal{L}_\eta f](x) = x^\eta \frac{1}{2\pi i} \int_\sigma \frac{\Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s)}{\Gamma(\alpha-\beta\eta+\beta s))\Gamma(1-\alpha+\beta\eta-\beta s)} f^*(s) x^{-s} ds. \tag{78}$$

The first representation of the operator (78) (a standard one) is obtained from the formula (49):

$$[\mathcal{L}_\eta f](x) = \begin{cases} x^\eta [P_{1/\beta}^{\alpha-\beta\eta,\beta\eta} I_{1/\beta}^{-\alpha,\beta\eta} f](x), & \eta > 0, \\ x^\eta [D_{1/\beta}^{-\alpha+\beta\eta,-\beta\eta} K_{1/\beta}^{\alpha,-\beta\eta} f](x), & \eta < 0. \end{cases}$$

To get another, more interesting representation of the operator (78) we use the supplement formula ($\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$) for the Euler Γ-function and obtain the following chain of the equalities:

$$\frac{\Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s)}{\Gamma(\alpha+\beta(s-\eta))\Gamma(1-\alpha-\beta(s-\eta))} = \frac{\sin(\pi(\alpha-\beta\eta+\beta s))}{\sin(\pi(\alpha+\beta s))}$$

$$= \frac{\cos(\pi\beta\eta)\sin(\pi(\alpha+\beta s)) - \sin(\pi\beta\eta)\cos(\pi(\alpha+\beta s))}{\sin(\pi(\alpha+\beta s))} = \cos(\pi\beta\eta) - \sin(\pi\beta\eta)$$

$$\times \frac{\cos(\pi(\alpha+\beta s))}{\sin(\pi(\alpha+\beta s))} = \cos(\pi\beta\eta) - \sin(\pi\beta\eta)\frac{\Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s)}{\Gamma(1/2+\alpha+\beta s)\Gamma(1/2-\alpha-\beta s)}.$$

Substituting the last representation into (78), we have

$$[\mathcal{L}_\eta f](x) = x^\eta(\cos(\pi\beta\eta)f(x) - \sin(\pi\beta\eta)[\mathcal{H}_\beta^\alpha f](x)), \tag{79}$$

where

$$[\mathcal{H}_\beta^\alpha f](x) = \frac{1}{2\pi i}\int_\sigma \frac{\Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s)}{\Gamma(1/2+\alpha+\beta s)\Gamma(1/2-\alpha-\beta s)} f^*(s)x^{-s}ds \tag{80}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{f(xt^\beta)t^{-\alpha}}{t-1}dt$$

is a generalized Hilbert transform (the integral in the right side of relation (80) is considered in the sense of principal value). The last representation follows from the Parseval formula (32), the relation (20) and the shift property of the Mellin integral transform (13). The case $\alpha = 0, \beta = 1$ of the operator (80) corresponds to the classical Hilbert transform (48).

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Dept. of Business Informatics
European University Frankfurt (Oder)
P.O. Box 1786
15207 Frankfurt (Oder) – GERMANY

e-mail: loutchko@uni-ffo.de

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